Measures of Consonances in a Goodness-of-fit Model for Equal-tempered Scales

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Abstract

In this paper a general model is described which measures the goodness of equal-tempered scales. To investigate the nature of this 'goodness', the consonance measures developed by Euler and Helmholtz are discussed and applied to two different sets of intervals. Based on our model, the familiar 12-tone equal temperament does not have an extraordinary goodness. Others, such as the 19-tone equal temperament look as least as promising. A surprising outcome is that when intervals from the just minor scale are chosen to be approximated by an n-tone equal temperament system, good values for n are $9, 22, 27$ and 36, rather than the commonly used $n = 12$.

1 Introduction

Many different tuning systems have been developed in the past. Nowadays, for western music, all keyboard instruments are tuned in equal temperament where the octave is divided into 12 equal parts.

Other musicians such as singers or string players, strive to play in what is called 'just intonation'. This relies on the idea that two tones sound best for the ear if they have a simple frequency ratio.

It is well known that for keyboard instruments, it is not possible to tune to just intonation. For these instruments a tuning system has to be developed which approaches just intonation as well as possible. In our equal temperament system the octave is perfect and the fifth is approximated very closely. Since these two intervals are in general judged as the most important ones, and the rest of the intervals can be matched to the basic intervals from just intonation by an approximation acceptable for the ear, this temperament system is considered to be a good tuning system.

However, several people have investigated whether this temperament system could be improved such that more ratios from just intonation can be approximated more closely. One way to do this is to create an equal temperament system with another division than 12 tones per octave. In the last decades several microtonal systems have been constructed and explored (Hall 1988; Krantz and Douthett 1994; Krantz and Douthett 2000). To investigate what would be a suitable number of parts to divide the octave into, one has to find out which frequency ratios, in which order of importance, should be approximated.

The rest of this paper is organized as follows: In Section 2, two different sets of intervals appearing in music are discussed. To order these intervals, Euler's Gradus function and Helmholtz' roughness function are used. Section 3 describes a goodness-of-fit model that we use to investigate which equal tempered system best approximates a set of ratios from just intonation. The input for this model is given by the four sets of ordered intervals resulting from Section 2. In Section 4 we discuss the values of goodness according to our model and describe a special case to show that 12-tone equal temperament it is not always satisfying. We end with some conclusions in Section 5.

2 Measures of consonance

Just intonation is defined as any system of tuning in which all intervals can be represented by ratios of whole numbers, with a strongly implied preference for the smallest numbers compatible with a given musical purpose (Other Music, Inc.). In this paper, our choice of preference will be based on notions of consonance.

Certain intervals are perceived as being more consonant than others. In approximating just intonation by an equal temperament, choices have to be made about which intervals have priority to approximate. In order to let resolution in music have the greatest effect, the most consonant intervals appearing in music should be approximated most closely.

Therefore, two questions are to be addressed: 1) which intervals appear naturally in music, and 2) what is the 'order of consonance' of these intervals? In answering the first question, we could e.g. take the intervals occurring in the major scale in just intonation. The just major scale is defined as the scale in which each of the major triads I, IV and V is taken to have frequency ratios 4:5:6 (for example the frequencies of do, mi, so, are rated as 4:5:6). Table 1 shows the ratios of the notes in the just major scale compared to the fundamental. Now all intervals coming from this scale (with respect to every other note) can be seen as the

Note	do	re	mi	fa	so	la	ti	do
Ratio	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Table 1: Frequency ratios between the different notes of the major scale and the fundamental 'do'.

S_1	S_2
octave (2/1)	octave (2/1)
fifth(3/2)	fifth (3/2)
fourth (4/3)	fourth (4/3)
major third (5/4)	major third (5/4)
minor third (6/5)	minor third (6/5)
major sixth (5/3)	major sixth (5/3)
minor sixth (8/5)	minor sixth (8/5)
major whole tone (9/8)	major whole tone (9/8)
minor whole tone (10/9)	minor seventh (9/5)
major seventh (15/8)	sub minor seventh (7/4)
minor seventh (9/5)	sub minor third (7/6)
diatonic semitone (16/15)	super second (8/7)
augmented fourth (45/32)	sub minor fifth (7/5)
diminished fifth (64/45)	super major third (9/7)
subdominant minor seventh (16/9)	
Pythagorian minor third (32/27)	
Pythagorian major sixth (27/16)	
grave fifth (40/27)	
acute fourth (27/20)	

Table 2: Set S_1: intervals coming from the just major scale, and set S_2: intervals appearing in the harmonic series up to the ninth harmonic. The names of the intervals are taken from Helmholtz (1954).

set of intervals most common in Western music. Let us call this set S_1 (see Table 2).

Another possibility is to take the intervals appearing in the harmonic series within a certain number of harmonics. Here we will consider the first nine harmonics. A reason to choose this set is to also include intervals involving 7 in their ratio[1]. Let us call this set S_2 (see Table 2).

What is the order of consonance of a set of intervals? As mentioned, there is a preference for lower numbers. But how to compare ratios like $6/5$ and $7/4$? For several ratios there is no consensus. Therefore, no unique function to describe the order of consonance exists.

2.1 Euler's Gradus function

Euler developed a Gradus function Γ which applies to whole frequency ratios x/y from just intonation (Euler 1739). The function is defined as a measure of the simplicity of a ratio. Applied to the problem of consonance this means that the lower the value $\Gamma(x/y)$ the simpler and the more consonant the interval.

Any positive integer a can be written as a unique product $a = p_1^{e_1} \cdot p_2^{e_2} \dots p_n^{e_n}$ of positive integer powers e_i of primes $p_1 < p_2 < \dots < p_n$. Euler's formula is now defined as:

$$\Gamma(a) = 1 + \sum_{k=1}^{n} e_k(p_k - 1) \qquad (1)$$

and for the ratio x/y the value is $\Gamma(x \cdot y)$. According to this formula, the order of consonance for set S_1 and set S_2 is given in Table 3.

S_1	S_2
2/1	2/1
3/2	3/2
4/3	4/3
5/4 , 5/3	5/4, 5/3
6/5, 9/8, 8/5	6/5, 9/8, 8/5
16/9	7/4
10/9, 9/5, 15/8	7/6, 8/7, 9/5
16/15, 27/16	7/5, 9/7
32/27	
27/20	
45/32, 40/27	
64/45	

Table 3: Order of consonance for ratios of set S_1 and S_2 according to Euler's Gradus function, from most to least consonant.

Note that there is not a perfect match with musical intuition. For example the major whole tone (in music considered to be a dissonant tone) is placed on the same level as the minor third.

2.2 Helmholtz' roughness function

Helmholtz defined the roughness of an interval between tones p and q on the basis of the sum of beat intensities $I_n + I_m$ associated with the n^{th} harmonic of p and the m^{th} harmonic of q (Helmholtz 1954). This roughness depends on the ratio n/m, but also on the intensity of the harmonics (and therefore on the type of sound) and on the register of the tones (in lower positions, intervals tend to sound more rough). Helmholtz calculated the roughness of intervals in the c'-c" octave, and based the intensity of the harmonics on violin sound. For the formal definition we refer to Helmholtz (1954). The order of consonance of set S_1 and S_2 according to this roughness function is given in Table 4[2].

Again, one would expect this ordering to coincide with a musical intuition, but an interval known as dissonant (augmented fourth $45/32$) is placed on the same level as two consonances. Note also that the fifth and the octave are judged to be equally consonant. This ordering according to Helmholtz differs from the ordering in Table 3. Remarkably, for both measures, the preference for lowest numbers is not entirely followed.

3 Goodness-of-fit model

Several functions have been defined to measure the goodness of a given n-tone equal-tempered scale (Hall 1988; Krantz and Douthett 1994). With our goodness-of-fit model we want to measure which equal temperament best approximates ratios from just intonation.

[1] This can be useful in tuning chords. For example, a dominant seventh chord is sometimes to be tuned as $4:5:6:7$.

[2] Since Helmholtz (1954) did not consider the interval ratio $16/9$ it is not included in Table 4.

S_1	S_2
2/1, 3/2	2/1, 3/2
4/3	4/3
5/3	5/3
5/4	5/4
6/5, 8/5, 45/32	7/4
16/9	6/5, 8/5
27/16	9/5, 7/6, 7/5
9/5	8/7
32/27	9/8
27/20	
64/45	
9/8	
10/9	
15/8	
40/27	
16/15	

Table 4: Order of consonance for ratios of set S_1 and S_2 according to Helmholtz's roughness function, from most to least consonant.

Given an n-tone equal temperament (the octave divided in n equal parts), the ratio R is best approximated when

$$|\log_2 R - \frac{m}{n}| \quad (2)$$

is as small as possible, where m is an integer. The number of steps m in an n-tone scale that minimizes (2) is

$$m = \text{int}(n \log_2 R + 0.5), \quad (3)$$

where $\text{int}(x)$ is the integer part of x. With (3) substituted in (2), an error function E is now defined:

$$E(R, n) = |\log_2 R - \frac{1}{n}(\text{int}(n \log_2 R + 0.5))|. \quad (4)$$

This is already a measure of the goodness of an n-tone scale for a ratio R. Since the function applies to equal temperaments, it necessarily yields the same values for an interval and its inverse (for example a fifth 3/2 and a fourth 4/3). Note that there is always a better fit possible if a higher value for n is chosen. But, for reasons of pitch discrimination, high values for n are usually rejected. Since we rather want to obtain a high value from our function when the fit is good and a low value when the fit is bad, and since we want to make the difference between the fits more visible, we take the logarithm of E

$$-\log_{10} E(R, n) \quad (5)$$

as final error function. Different ratios have to be fit simultaneously and weighted according to consonance, therefore we introduce a weight p_i for each ratio R_i, such that

$$\sum p_i = 1. \quad (6)$$

The final expression of the goodness-of-fit function used for the evaluation of n-tone temperaments is:

$$f(n) = \sum_{i=1}^{m} p_i(-\log_{10} E(R_i, n)). \quad (7)$$

4 Results

We have calculated the values of f for the two sets of intervals and Euler's or Helmholtz' order of consonance, for $n \in [1, 55]$. We chose weights for the ratios inversely proportional to the values of Euler's Gradus function or Helmholtz' roughness function. The results are shown in Figures 1 to 4.

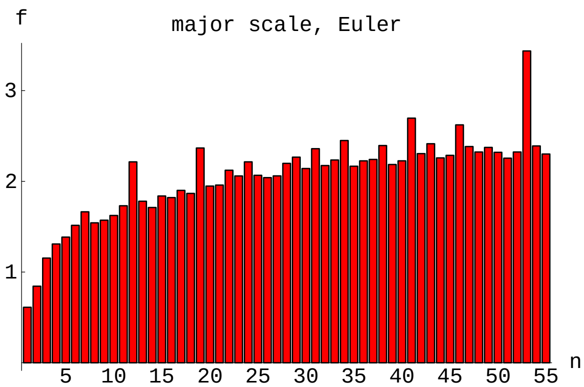

Figure 1: Values of f calculated from the set intervals of the major scale combined with Euler's measure of consonance.

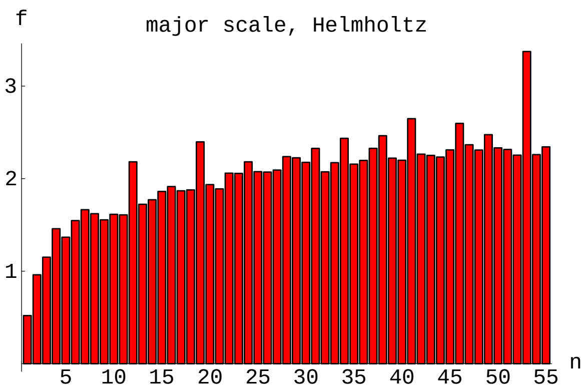

Figure 2: Same as Figure 1 but now with Helmholtz' measure of consonance.

All figures show peaks for $n = 12, 19, 41, 53$ and a couple of other peaks occur depending on the input. We notice that $n = 12$, the familiar 12-tone temperament doesn't look like a special case, $n = 19$ is as least as promising. Although Euler's and Helmholtz' functions are quite different in judging the consonances, the outcome of our goodness-of-fit model is rather similar. This is mainly due to the model's property that judges inverse ratios in the same way. To further investigate the consequence of different measures of consonances a 'non-equal temperament system' will be used in future research.

4.1 Special case: minor mode

To test an n-tone equal-tempered system, we chose the interval set coming from the just major scale and the set appearing in the harmonic series up to the ninth harmonic. Other choices are possible as well. For example, which intervals should we choose for a minor

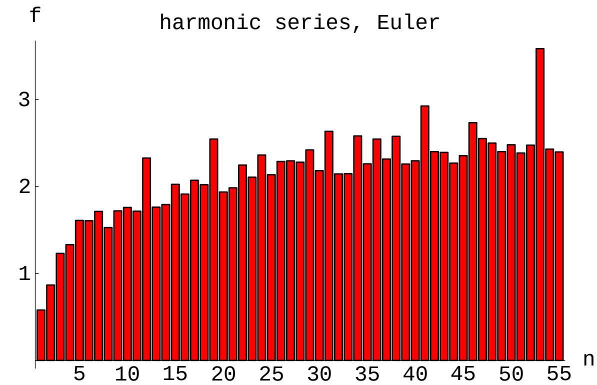

Figure 3: Values of f calculated from the set intervals from the harmonic series combined with Euler's measure of consonance.

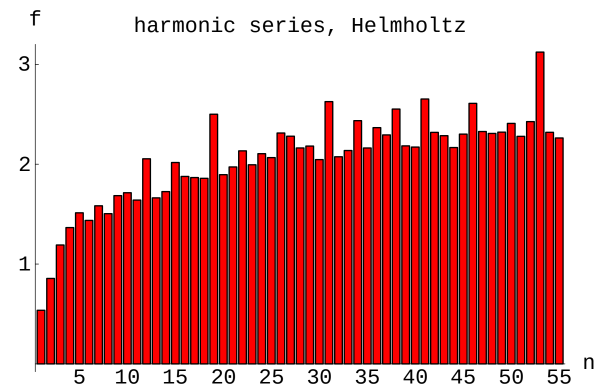

Figure 4: Same as Figure 3 but now with Helmholtz' measure of consonance.

scale? As the just major scale is built from three major triads with frequency ratios 4:5:6, a minor scale is built from three minor triads. The lowest numbers producing a minor triad in the harmonic series are 6:7:9 [3] (McIntyre 2002). One could argue that these are the most important ratios in the minor scale and should therefore be weighted most. If we do this, Figure 5 is the result. Totally different peak values appear compared to previous figures. We notice high values for $n = 9, 22, 27, 36$ while $n = 12$ is not higher than average. According to these results a piece of music in minor mode would not sound particularly good in 12-tone equal-temperament. We suggest that this observation should be followed up by psychological research to evaluate its cognitive reality.

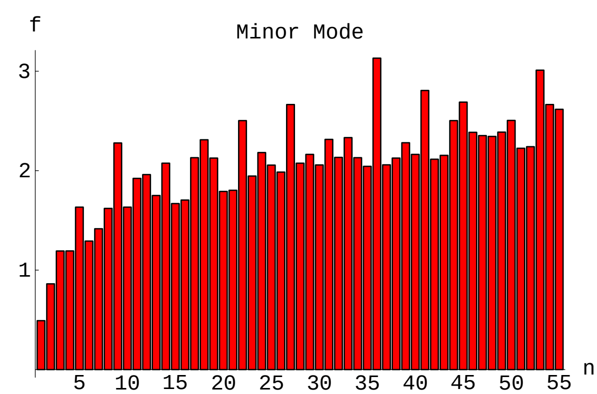

Figure 5: Values of f calculated from the set of intervals: $6 : 7$, $7 : 9$ and $2 : 3$, with equal weighting.

[3] However, most people choose the ratios $10 : 12 : 15$ for a minor triad, because in this way it is built from the same major and minor third used in a major triad.

5 Conclusion

In this paper we have checked how well several equal temperament systems approximate the frequency ratios from just intonation. As test sets, we chose intervals from the just major scale and intervals from the harmonic series. To weight the intervals, Euler's Gradus function and Helmholtz' roughness function have been used. It turns out that for these cases a division of the octave in $12, 19, 41$ or 53 would be a good choice. Although all relevant figures show peaks for these values, these peaks are not the only ones. Depending on what set and measure of consonance used, peaks for other divisions of the octave exist as well. Taking intervals according to the minor mode confirms that a good choice for a division of the octave depends highly on the set of intervals used. However, major and minor are just a selection from a variety of possibilities.

To conclude, we want to stress that testing an n-tone equal-tempered system involves choosing a set of intervals and a measure of consonance. These choices can again depend on a type of music, a special purpose or taste. These results will hopefully trigger new psychological research for investigating the cognitive reality of different tone scales especially with respect to the minor mode.

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