# Non-linear guitar body models

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**Abstract**

*This paper describes a non-linear model for the body of an acoustic guitar. The body is modeled using a linear model for the principal modes, and a static, saturating non-linearity, determined using a sequence of impulses of growing amplitude.*

## 1 Introduction

The guitar body acts as an acoustic amplifier for the vibrations of the strings. It ensures that enough energy is reflected to the strings to sustain the note and that a note of significant volume is audible to the listeners. The shape and materials used are determining factors of the timbre and the spatial sound radiation pattern of the instrument. A guitar body is a complex structure consisting of thin, more or less flexible plates, bracings, and an enclosed air cavity. The frequency response of a guitar body shows many resonances and anti-resonances. When mechanically driven at the bridge, the lowest resonance is usually a bar bending mode, but at too low a frequency to be excited by the vibration of real strings (Fletcher and Rossing 1991). Most guitars have three string resonances in the 100-200 Hz range due to coupling between the fundamental modes of the top and back plates, and the Helmholtz mode of the air cavity. At the lowest of the three resonances, the top and back plates move in opposite directions. The guitar top vibrates in many modes; those of low frequency bear considerable resemblance to those of a rectangular plate when no bracings are present. The bracings are necessary to add mechanical rigidity to the top plate, and have a strong impact on the vibration modes of the top plate. The higher air cavity modes resemble standing waves in a rectangular box. Disregarding the ribs and the back plate, the top-plate and air cavity can be modeled as two coupled oscillators. Adding the back plate yields a three-oscillator model. In most guitars, the addition of the back results in a downward shift of the two primary resonances (Christensen 1982), (Meyer 1974), (Rossing, Popp, and Polstein 1985). The higher two resonances usually occur around 200 Hz, depending upon the stiffness of the the top and back plates. The motion of the air and the top plate is in the same direction, thus resulting in strong sound radiation. The resonances of the top plate, back plate, and air cavity generally combine to give at least one strong resonance around 300 Hz in a classical guitar, but closer to 400 Hz in a cross-braced folk guitar. Above 400 Hz, the coupling between top and back plate appears to be relatively weak, and the observed resonances are due to resonances in one of the plates. There is a definite link between the subjective quality of a guitar and its frequency response (Meyer 1983).

Guitar body models play an important role in physical modeling sound synthesis, either as an equalization filter or as an integral part of the model (Nackaerts et al. 2001). Most studies focus on linear modeling techniques, as these are able to model the major part of the body response. There are however indications that a guitar body is not entirely linear. This is backed by subjective descriptions by guitar builders on the influence of the wood used.

## 2 Linear models

Linear body models have been well studied in the past. For sufficiently small input signals, the body of a guitar is a linear system, and can be modeled using all the conventional linear modeling methods. The impulse response of a guitar body excited at the bridge shows a large number of poles. It has been shown that more than 300 poles are needed to obtain a model that is indistinguishable from the original by the listener (Penttinen, Karjalainen, Paatero, and Järveläinen 2001).

### 2.1 FIR and IIR+FIR models

The easiest way of modeling the guitar body is by using the complete impulse response of the body as an FIR filter. The basis functions used in FIR filters are the orthonormal functions $z^{-k}$. A FIR model of a system $G(z) \in H_2$ ($H_2$ denoting the space of stable, strictly-proper transfer functions) consists of a finite number of expansion terms and takes on the form

$$\hat{G}(z) = \sum_{k=1}^{N} g(k) z^{-k}, \qquad (1)$$

where $g(k)$ are impulse response coefficients. If the decay rate of the signal is low compared to the sampling rate, very high order FIR filters are needed. The computational cost is quite high, especially when using a sample-oriented method. When block-processing is used, several fast-convolution frequency domain techniques can be used to speed up the computation. The long FIR size of several tens of thousands of samples is the main bottleneck of using this method in the fully coupled model, as this model does not easily allow block processing. The size of the FIR filter can be significantly reduced by using IIR resonators to model the first few body resonances, and using the FIR filter only for the residual signal. The design of the IIR filterbank is straightforward. The peak frequency is best found in the frequency domain, by quadratic interpolation of the peak in the spectrum. In this case, the damping is quite high, so the peaks are wider, making the manual determination of the 3dB bandwidth a possibility. Methods using the Short Time Fourier Transform are less interesting, exactly because of the higher damping, making the signals too short in time for an accurate measurement of the exponential slope. A better approach in this case is to use an optimization technique to minimize the cost function

$$\mathcal{E} = \sum W(\mathcal{S} - \mathcal{H}_{\text{peak}})^2, \qquad (2)$$

where $\mathcal{S}$ is the spectrum of the body impulse response, $\mathcal{H}$ the response of the IIR filter, and $W$ a windowing function around the peak. The residual signal is calculated by either a time-domain subtraction of the output of the IIR resonator, or by filtering the body impulse response with a notch filter. The procedure is then repeated for the other modes, until the residual signal is short enough to be used as a FIR filter.

In general, the guitar body is a linear time invariant model, and could be described using the standard LTI model structures. The complete model structure is given by

$$A(z)y_k = \frac{B(z)}{F(z)}u_k + \frac{C(z)}{D(z)}e_k, \qquad (3)$$

where $e_k$ is Gaussian white noise. We assume that the recording of the body impulse only adds white noise to the output and obtain the output error (OE) model structure

$$y_k = \frac{B(z)}{F(z)}u_k + e_k. \qquad (4)$$

As was determined before, the order of $B(z)/F(z)$ is approximately 350. The model parameters can be found with either prediction error methods or correlation methods.

Prediction error methods are based on the minimization of the prediction error sequence $\hat{e}_k$

$$\hat{e}_k = y_k - G(z,\theta)u_k, \qquad (5)$$

with $G(z,\theta) = B(z)/F(z)$. This is achieved by optimizing the parameter vector$\theta$ with as a cost function

$$V(\theta) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{2}(\hat{e}_k^f(\theta))^2, \qquad (6)$$

where $\hat{e}_k^f(\theta)$ is the error sequence, filtered to stress a certain part of the spectrum. The frequency domain interpretation of the prediction error method shows that this is just a generalization of the cost function (2) or

$$V(\theta) = \int_{-\pi}^{\pi}\frac{1}{2}|G_0(e^{j\omega}) - G(e^{j\omega},\theta)|^2|L(e^{j\omega})|^2|U(\omega)|^2 d\omega, \qquad (7)$$

with $G_0(z)$ the real transfer function, $G(z,\theta)$ the model, $U(\omega)$ the spectrum of the input, and $L(z)$ the filter used to filter the prediction error. For our purposes, instead of determining one high-order model, we calculate a set of low-order models. Each low order model is obtained by band-pass filtering the prediction error around the previously selected peaks in the spectrum. If the input is a true impulse, this method simplifies to equation (2). When calculating a second order model for each subband, we obtain

$$y_k = \sum_{i=1}^{N}\frac{b_0 + b_1z^{-1} + b_2z^{-2}}{a_0 + a_1z^{-1} + a_2z^{-2}}u_k + e_k, \qquad (8)$$

with $N$ the total number of bands considered. This procedure results in a parallel set of IIR filters.

The practical determination using these conventional system identification methods require long, persistent excitation signals. Therefore, a synthetic output is generated by filtering white noise with the complete impulse response data as FIR coefficients. The resulting longer input-output signals are used during identification.

## 2.2 Kautz filters

Kautz filters (Kautz 1954), (Wahlberg 1991b), (Wahlberg 1991a), (Hoog 2001) are fixed-pole IIR filters structurally organized to produce orthonormal tap-output impulse responses. The transversal Kautz filter can be seen as a generalization of FIR and Laguerre filter structures, providing IIR-like parametric modeling combined with the superior numerical properties of FIR filters (Paatero and Karjalainen 2001), (Penttinen et al. 2001).

Taking into account the high FIR filter order needed for accurate modeling, one could use a different orthonormal basis of $H_2$. Let the functions $F_k(z)$ with $k \in \mathbb{N}$ denote the basis elements of such a general basis. Then the transfer function $G(z) \in H_2$ can be expanded as

$$G(z) = \sum_{k=1}^{\infty} c_k F_k(z), \qquad (9)$$

The aim is to choose the basis $\{F_k(z)\}_{k\in\mathbb{Z}}$ such that the expansion coefficients $c_k$ rapidly converge to zero. A straightforward approach to this problem is to ortho-normalize the set of functions

$$f_{i,j}(z) = \frac{1}{(z-a_i)^j}, \tag{10}$$

where the poles $a_i$ can in principle be any complex number with modulus smaller than one, and $i \in \mathbb{N}, 1 \le j \le m_i$. Applying a Gram-Schmidt procedure to the sequence of functions $f_{i,j}(z)$ yields the orthonormal functions. If the functions are constrained to be real-rational functions, one gets the Kautz construction equations. A structure based on the Kautz basis functions that is well-suited for implementation as a transversal-like filter is

$$\phi_k(z) = a_k \frac{z\sqrt{1-|\beta_k|^2}}{1-\beta_i z} \prod_{i=0}^{k} \frac{1-\bar{\beta}_i z}{z-\beta_i}. \tag{11}$$

For an efficient model, a good set of poles should be selected. The determination of the poles is however not a trivial problem, and several design procedures can be found in the literature (Brinker, Benders, and e Silva 1996). The lowest-order models are obtained when the basis function poles correspond to the system poles. In the case of a guitar body model, we know the dominant poles, and can use these as the poles to build the Kautz functions.

# 3 Non-linear models

The body of a guitar is only linear for small excitations. For larger amplitudes, the wooden top plate tends to saturate the output due to its limited flexibility and bracings. The effect can be measured using series of impulses with increasing amplitude.

Wiener-Hammerstein systems consist of two linear blocks and a static non-linearity, shown in figure 1. We assume that in the case of a guitar body the input is saturated, yielding a Wiener system. The output of the Wiener-Hammerstein is

$$x_2(t) = g(x_1(t)), \tag{12}$$
$$U(s) = H_1(s)X_2(s), \tag{13}$$

The static non-linearity is described by a polynomial function

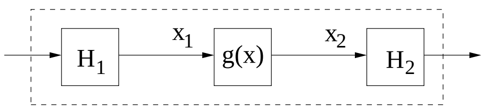


Figure 1: A complete Wiener-Hammerstein system with two linear blocks and a static non-linearity.

$$g(x) = \sum_{k=1}^{N} b_k x^k. \tag{14}$$

The traditional non-linear system identification techniques often require specific input, like a multi-sine excitation. This requires specialized equipment. Assuming that the body is linear for small excitations, the linear part can be estimated for low-amplitude input. If the system is purely linear, amplification of a low-amplitude impulse response will approximately yield the high-amplitude impulse response. The static non-linearity will however distort the signal. The difference between the expected linear output and the recorded output for high-amplitude excitation can be exploited to determine the non-linearity.

The coefficients of the non-linear function $g(x)$ are easily determined as the solution of

$$\mathbf{A}b = C, \tag{15}$$

where $b$ is the vector of the coefficients $b_{k=1\ldots N}$, and $C$ the large-excitation signal, and with the matrix $A$

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^N \\ 1 & x_2 & x_2^2 & & x_2^N \\ \vdots & & & & \vdots \\ 1 & x_j & x_j^2 & \ldots & x_j^N \end{bmatrix}, \tag{16}$$

constructed using the amplified small-excitation impulse. Solving the system in the least-squares sense yields the coefficients.

A fairly similar method consists of first building a static input-output relationship. First, two sequences of impulses of the same amplitude are recorded and the expected responses are calculated. The expected values are then placed in amplitude bins, and the input-output relation is determined for matching bins. A polynomial approximation can then be found. This second method yields approximately the same results, and allows the weighting of measurements, and an estimation of the error. The methods have been validated using synthetic datasets.

Using this method, the behavior of several guitars bodies has been analyzed. It appears that some guitars exhibit a higher distortion/compression at higher amplitudes. Figure 3 shows a polynomial approximation of the non-linearity of a Taylor 514CE steel string guitar. The dots on the figure indicate the input-output relation between the amplified low-amplitude impulse response, and the high-amplitude impulse response. Each dot on the figure represents thousands of samples for the lower amplitudes to about hundred samples at high amplitude. We observe a saturation characteristic at higher amplitudes.

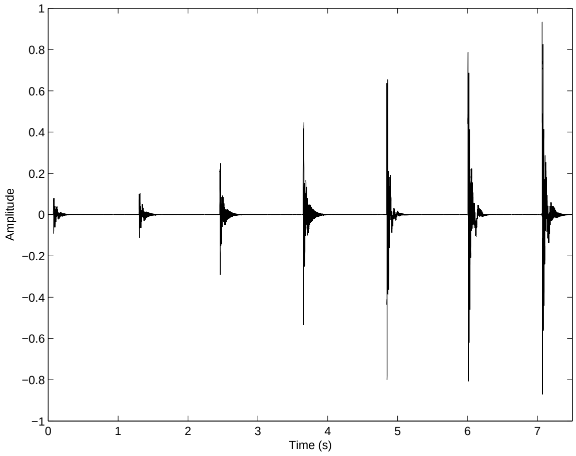


Figure 2: The recorded series of impulses with growing amplitude.

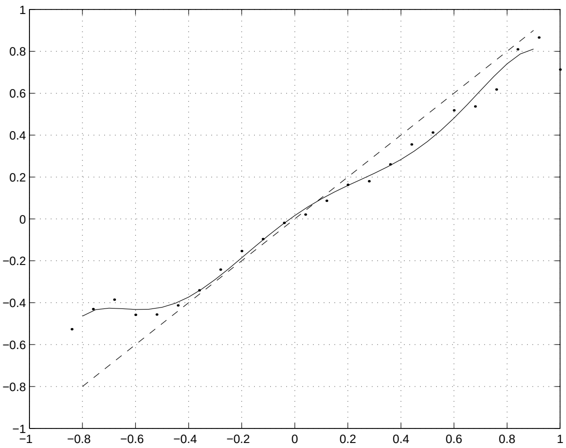

Figure 3: This figure shows the static non-linearity added to the body model. The dashed line shows a linear input-output relation. The dots represent the measured input-output relations for a given amplitude, and the solid line is a polynomial fit. The compression effect is visible for the negative input values. This means that above a specific playing level, the output amplitude will only slightly increase, and the sound will be distorted. This effect depends on the type of wood used and is well known by guitar players.

# 4 Conclusion

The body of a guitar can be modeled using linear techniques combined with a static non-linearity. The parameters of the resulting Wiener system were identified using a sequence of impulses. By assuming that the system is linear for small excitations, the input-output relation of the static non-linearity can be build, using either the solution of an overdetermined system, or by using a statistical method. Measurement of the behavior of different guitars has shown that there is a slight compression at higher amplitudes.

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